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Some Matrix Aspects of Generalized Dimensional Analysis*

M. J. MORAN

Department of Mechanical Engineering, The Ohio State University, Columbus, Ohio, USA

K. M. MARSHEK

Department of Mechanical Engineering, The University of Connecticut, Storrs, Connecticut, USA

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SUMMARY

Recent group theoretic techniques effective for reducing the number of variables in systems of partial differential equations are invoked to formulate a generalized approach to dimensional analysis. It is then shown that significant conclusions can be elicited upon consideration of a *dimensional matrix* arising via the group formulation. In particular, a generalized form of Sedov's self-similarity criterion is shown to exist within the framework of the approach developed. The presentation is concluded by consideration of three illustrations.

1. Preliminaries

One of the first systematic applications of group theory to the dimensional-similarity analyses of problems arising in fluid mechanics and related engineering specialities is found in Birkhoff's classic, *Hydrodynamics* [1]. The author remarks in this volume, "...I believe... we have only begun to explore the applications of the group concept to differential equations." The correctness of this observation is amply attested by the numerous contributions made in this area during the past two decades: [2], [3], [4], [5], and still others might be mentioned.

A central feature of dimensional-similarity analyses via group theory is the use of a group of continuous r-parameter transformations,

$$\bar{z}_i = f_i(z_1, z_2, \ldots, z_n;\ A_1, A_2, \ldots, A_r) \qquad (i=1, \ldots, n) \tag{1.1}$$

and the *absolute invariants* associated with the group: functions π which satisfy, $\pi(z_1, z_2, \ldots, z_n) = \pi(\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n)$; (see [6] for an excellent introduction to the theory of continuous transformation groups). The A's of (1.1) are the group parameters, of which more is said later. The variables z_i of (1.1) correspond to be variables appearing in the set of governing equations under consideration—i.e., correspond to the variables of a set of differential equations and its associated set of boundary and/or initial conditions.

The *first step* in the application of the group theoretic approach to dimensional-similarity analyses is the establishment of a group under whose transformations the set of governing equations is *invariant in form*. Next, the absolute invariants of the group are used to express the governing equations in terms of *fewer* variables, this being the objective of many dimensional-similarity analyses.

The establishment of an appropriate group is frequently the crucial step. Considerable effort has been directed, therefore, to the problem of devising means for establishing a group under whose transformations a *given* set of governing equations is invariant in form. An extremely powerful approach to this problem would clearly be a method by which an investigator could begin an analysis by considering the class of groups (1.1), and then seek in a systematic manner restrictions imposed upon the functions f_i by a given set of governing equations in order that the equations be invariant in form. Certain investigators have proposed systematic methods aimed

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at this objective, [4], [5], [7]; but while effective, the methods often achieve generality at the expense of considerable manipulation.

Experience reveals that for a wide range of engineering applications a sufficiently general class of groups to initiate an analysis is provided by the *subclass* of r-parameter groups (1.1) with the form,

$$\bar{z}_i = C_i(A_1, \ldots, A_r) z_i + D_i(A_1, \ldots, A_r) \qquad (i=1, \ldots, n) \tag{1.2}$$

see [8], [9]. The initial objective of such analyses is to establish restrictions imposed upon the functions C_i, D_i by a given set of governing equations in order that the set be invariant in form. The advantage of initiating an analysis with a class of groups (1.2) is that a significant reduction in manipulation is often realized over that required by the above-mentioned systematic methods; but while there may be less manipulation, the approach may in some cases lead to less general results than would be derived via the more inclusive class (1.1).

Another, still more special subclass of (1.1) which has been found to have utility in a number of dimensional-similarity analyses is one of the form,

$$\bar{z}_i = A_1^{\gamma_{i1}} \ldots A_r^{\gamma_{ir}} z_i \qquad (i=1, \ldots, n). \tag{1.3}$$

Many of the manipulative difficulties inherent in the foregoing approaches are eliminated by initiating an analysis with a class of groups with the form (1.3): With (1.3) the initial objective is merely to establish restrictions imposed upon the exponents $\gamma_{i\alpha}$ ($\alpha=1, \ldots, r$) by a given set of governing equations in order that the set be invariant in form. The expense of this manipulative ease, however, may well be a loss of some generality. This point notwithstanding, the focus of the present paper is upon the utility of r-parameter groups (1.3) for dimensional-similarity analyses. Of special interest are the conclusions that may be established merely by considering the associated matrix of exponents,

$$\gamma: [\gamma_{i1}, \ldots, \gamma_{ir}].$$

Following Bridgman [10], it is helpful to introduce,

Definition 1: *Relative to the group* (1.3), *the dimensions of the variable denoted by* z_i *are given by the r-tuple* $(\gamma_{i1}, \ldots, \gamma_{ir})$.

The matrix γ associated with a group (1.3) will be termed, therefore, a *dimensional* matrix.

2. Generalized Dimensional Analysis

Groups with the form (1.3) bear a close relationship to *traditional* approaches to dimensional analysis; e.g., see [1]. As a concrete example, consider a flat plate which is immersed in an incompressible, viscous fluid, and which is accelerated from rest to a constant plate velocity $U>0$; mathematically [11],

$$u_t - \nu u_{yy} = 0 \quad \text{(momentum)} \tag{2.1}$$

subject to,

$$\begin{aligned} &u \to 0 \quad \text{as } t \to 0 \quad \text{when } y \geqq 0 \\ &u \to 0 \quad \text{as } y \to \infty \quad \text{when } t \geqq 0 \\ &u \to U \quad \text{as } y \to 0 \quad \text{when } t > 0 \end{aligned} \tag{2.2}$$

where u denotes the fluid velocity parallel to the plate; $\nu (\nu>0)$ symbolizes the constant kinematic viscosity of the fluid; y represents position normal to the plate, and t signifies time. Finally, letter subscripts denote partial differentiation.

The conventional dimensional approach to this problem would be to associate *dimensional formulae* with each of the significant quantities, [10], [11]. Thus,

$$[u], [U]: L^{+1}\tau^{-1}, \; [y]: L^{+1}\tau^{0}, \; [t]: L^{0}\tau^{+1}, \; [\nu]: L^{+2}\tau^{-1}, \tag{2.3}$$

wherein the brackets [ ] mean "the dimensions of." The formulae (2.3) may be regarded as shorthand expressions for the *scale change equations*,

$$\bar{u}=L^{+1}\tau^{-1}u,\ \bar{y}=L^{+1}\tau^{0}y,\ \bar{t}=L^{0}\tau^{+1}t,\ \bar{v}=L^{+2}\tau^{-1}v,\ \bar{U}=L^{+1}\tau^{-1}U\,, \tag{2.4}$$

e.g., see [13], [14]. Equation (2.4) has a greater significance, however, than merely for changing scale; that is, (2.4) constitutes a two-parameter group, with the scale factors L and τ playing the role of group parameters. [The relationship of (2.4) with (1.3) is perhaps most readily seen by rewriting (2.4) with $A_1\equiv L$, $A_2\equiv\tau$.]

Having indicated the relationship of r-parameter groups (1.3) to conventional dimensional notions, attention is now briefly focused upon a central feature of this paper; namely, each of the variables in any set of governing equations under consideration is regarded as being in one of three distinct categories; (i) dependent, (ii) independent, (iii) physical. Thus, as an elementary illustration, the variables appearing in (2.1)–(2.2), u, y, t, v, U, may be identified as follows. The fluid velocity u may be identified as the *dependent* variable; the position and time coordinates x, t may be identified as *independent* variables; and the quantities U, v may be identified as *physical* variables. The principal results of this paper have application, therefore, to those sets of governing equations for which it is possible to unambiguously divide the variables appearing therein into the above-mentioned categories.

In recognition of the foregoing three categories for variables, the class of r-parameter groups (1.3) can be written somewhat more explicitly. Thus, to be considered in the following discussions are r-parameter groups with the form,

$$\left\{\begin{array}{lll}\bar{Z}_j=A_1^{a_{j1}}\dots A_r^{a_{jr}}Z_j & (j=1,\dots,n\geqq 1) & (2.5)\\ \bar{X}_k=A_1^{b_{k1}}\dots A^{b_{kr}}X_k & (k=1,\dots,m\geqq 1) & (2.6)\\ \bar{Y}_e=A_1^{c_{e1}}\dots A_r^{c_{er}}Y_e & (e=1,\dots,p\geqq 0) & (2.7)\end{array}\right.$$

wherein the Z's are to be associated with the *dependent* variables of a set of governing equations, the X's are associated with the *independent* variables, and the Y's are associated with the *physical* variables. [As a concrete example, (2.5) corresponds to the first transformation of (2.4), (2.6) corresponds to the second pair of transformations appearing in (2.4), and (2.7) corresponds to the last pair of transformations in (2.4)].

Subsequent discussions reveal that the dimensional matrix associated with (2.5)–(2.7) plays an important role. To facilitate the presentation, then, let B denote the $(m\times r)$ matrix $[b_{k1},\dots,b_{kr}]$; and let C denote the $(p\times r)$ matrix $[c_{e1},\dots,c_{er}]$. Similarly, let BC denote the $([m+p]\times r)$ matrix,

$$BC:\begin{bmatrix} b_{k1},\dots,b_{kr}\\ c_{e1},\dots,c_{er}\end{bmatrix}$$

The matrix BC is assumed to have rank r★, while the matrix C has rank s, $s\leqq r$. Thus, the dimensional matrix associated with (2.5)–(2.7) has rank r.

As an additional means of facilitating the presentation, the rows of BC are assumed to be arranged so that,

(*i*) When $s=r$, the first r rows of C are linearly independent;

(*ii*) when $s<r$, the first s rows of C *plus* the last $[r-s]$ rows of B are linearly independent.

To illustrate the foregoing notions, consider again (2.4). By inspection, the matrices B, C and BC are given, respectively by,

$$B:\begin{bmatrix}1&0\\ \\ 0&1\end{bmatrix},\quad C:\begin{bmatrix}2&-1\\ \\ 1&-1\end{bmatrix},\quad BC:\begin{bmatrix}1&0\\0&1\\2&-1\\1&-1\end{bmatrix}$$

Also, for (2.4), $n=1$, $m=p=r=s=2$.

★ This condition is required for the group parameters to be essential; see [14] for further detail.

Having defined the class of r-parameter groups to be considered in this paper: (2.5)–(2.7), and having introduced some important aspects of the dimensional matrices associated with such groups, attention now turns to certain features of a generalized dimensional analysis approach developed in [14]. One of the principal results of [14] is summarized in Theorem 1, which is formulated here in terms of a r-parameter group (2.5)–(2.7).

Theorem 1: *If the function I_j is invariant in form under an r-parameter group* (2.5)–(2.7)—*i.e., if $Z_j = I_j(X_1, \ldots, X_m; Y_1, \ldots, Y_p)$ transforms to $\bar{Z}_j = I_j(\bar{X}_1, \ldots, \bar{X}_m; \bar{Y}_1, \ldots, \bar{Y}_p)$, then $Z_j = I_j(\ldots)$ is equivalent to a relationship in fewer variables,*

$$\Pi_j(Z_j, X_1, \ldots, X_m; Y_1, \ldots, Y_p) = F_j(\pi_1(X_1, \ldots, X_m; Y_1, \ldots, Y_p), \ldots, \pi_\delta(\ldots)) \tag{2.8}$$

*wherein $\delta = [m+p-r] > 0$**, *and $\{\Pi_j, \pi_1, \ldots, \pi_\delta\}$ are independent absolute invariants of* (2.5)–(2.7).

In the present discussion Theorem 1 plays the role of the so-called Pi Theorem of conventional dimensional analysis.

To apply Theorem 1 requires expressions for the absolute invariants of (2.5)–(2.7). While [14] provides expressions for the invariants, it is illuminating to consider the manner in which they may be established. By definition, $\pi(X_1, \ldots, X_m; Y_1, \ldots, Y_p)$ is an absolute invariant provided that under the transformations (2.6)–(2.7),

$$\pi(\bar{X}_1, \ldots, \bar{X}_m; \bar{Y}_1, \ldots, \bar{Y}_p) = \pi(X_1, \ldots, X_m; Y_1, \ldots, Y_p)\,. \tag{2.9}$$

Upon differentiation of (2.9) with respect to each of the parameters in turn,

$$\sum_{k=1}^{m} \frac{\partial \pi}{\partial \bar{X}_k}\frac{\partial \bar{X}_k}{\partial A_\alpha} + \sum_{e=1}^{p} \frac{\partial \pi}{\partial \bar{Y}_e}\frac{\partial \bar{Y}_e}{\partial A_\alpha} = 0 \qquad (\alpha = 1, \ldots, r)\,. \tag{2.10}$$

And with (2.6)–(2.7) it follows that,

$$\frac{\partial \bar{X}_k}{\partial A_\alpha} = \left[\frac{b_{k\alpha}}{A_\alpha}\right]\bar{X}_k\,, \quad \frac{\partial \bar{Y}_e}{\partial A_\alpha} = \left[\frac{c_{e\alpha}}{A_\alpha}\right]\bar{Y}_e\,. \tag{2.11}$$

Combining (2.10) and (2.11), a system of first order, linear partial differential equations evolves,

$$\sum_{k=1}^{m} b_{k\alpha}\bar{X}_k \frac{\partial \pi}{\partial \bar{X}_k} + \sum_{e=1}^{p} c_{e\alpha}\bar{Y}_e \frac{\partial \pi}{\partial \bar{Y}_e} = 0 \qquad (\alpha = 1, \ldots, r)\,. \tag{2.12}$$

According to the theory of first order, linear partial differential equations [15], (2.12) has $[m+p-r]$ independent solutions. It will now be shown that each of the independent solutions may be determined in the form,

$$\pi = [\bar{X}_1]^{\Gamma_1}\ldots[\bar{X}_m]^{\Gamma_m}[\bar{Y}_1]^{\gamma_1}\ldots[\bar{Y}_p]^{\gamma_p} = [X_1]^{\Gamma_1}\ldots[X_m]^{\Gamma_m}\ldots[Y_1]^{\gamma_1}\ldots[Y_p]^{\gamma_p} \tag{2.13}$$

Indeed, upon substitution of (2.13) into (2.12) and simplification, a linear system of ordinary equations is derived,

$$\sum_{k=1}^{m} \Gamma_k \begin{bmatrix} b_{k1} \\ b_{k2} \\ \vdots \\ b_{kr} \end{bmatrix} + \sum_{e=1}^{p} \gamma_e \begin{bmatrix} c_{e1} \\ c_{e2} \\ \vdots \\ c_{er} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.14}$$

Therefore, to determine the $[m+p-r]$ independent absolute invariants π needed to apply Theorem 1 requires only that $[m+p-r]$ independent solutions be established to (2.14); (also see [13; Chapter 3)].

A like procedure can be invoked to show that the absolute invariants Π_j of Theorem 1 may be established in the form,

$$\begin{aligned} \Pi_j &= \bar{Z}_j[\bar{X}_1]^{A_{j1}}\ldots[\bar{X}_m]^{A_{jm}}[\bar{Y}_1]^{\lambda_{j1}}\ldots[\bar{Y}_p]^{\lambda_{jp}} \\ &= [Z_j][X_1]^{A_{j1}}\ldots[X_m]^{A_{jm}}[Y_1]^{\lambda_{j1}}\ldots[Y_p]^{\lambda_{jp}} \end{aligned} \tag{2.15}$$

* The special case $\delta = 0$ is discussed in [14] and will not be given further consideration here.

wherein,

$$\sum_{k=1}^{m} \Lambda_{jk} \begin{bmatrix} b_{k1} \\ b_{k2} \\ \vdots \\ b_{kr} \end{bmatrix} + \sum_{e=1}^{p} \lambda_{je} \begin{bmatrix} c_{e1} \\ c_{e2} \\ \vdots \\ c_{er} \end{bmatrix} = - \begin{bmatrix} a_{j1} \\ a_{j2} \\ \vdots \\ a_{jr} \end{bmatrix} \tag{2.16}$$

With the foregoing preliminaries in hand, the principal results of this paper are presented in the following article.

3. Principal Results

The statement of Theorem 1 does not suggest a preferred form for the required absolute invariants. However, experience reveals that for practical applications of the theorem it is frequently good practice to establish the required set of absolute invariants in one of the two forms to be given in Theorem 2 and Theorem 3.

Theorem 2 treats the case where the rank r of the matrix BC associated with an r-parameter group (2.5)–(2.7) equals the rank s of the matrix C; the case $r>s$ is then considered in Theorem 3.

Theorem 2: *If, and only if $r=s$, the set of $[n+m+p-r]$ independent absolute invariants required by Theorem 1 may be obtained in the form**,

$$\Pi_j = Z_j[Y_1]^{\lambda_{j1}} \dots [Y_r]^{\lambda_{jr}} \qquad (j=1, \dots, n) \tag{3.1}$$

$$\pi_k = X_k[Y_1]^{\gamma_{k1}} \dots [Y_r]^{\gamma_{kr}} \qquad (k=1, \dots, m) \tag{3.2}$$

$$\tilde{\pi}_\rho = Y_\rho [Y_1]^{\delta_{\rho 1}} \dots [Y_r]^{\delta_{\rho r}} \qquad (\rho = [r+1], \dots, p)\,. \tag{3.3}$$

Eq. (3.1) is readily established via (2.15)–(2.16) upon utilizing the assumed condition that when $s=r$, the first r rows of the matrix C are linearly independent. Thus, (2.16) yields the following system of equations for the exponents $\lambda_{j\alpha}$ of (3.1),

$$\sum_{\alpha=1}^{r} \lambda_{j\alpha} \begin{bmatrix} c_{\alpha 1} \\ c_{\alpha 1} \\ \vdots \\ c_{\alpha r} \end{bmatrix} = - \begin{bmatrix} a_{j1} \\ a_{j2} \\ \vdots \\ a_{jr} \end{bmatrix} \qquad (j=1, \dots, n)\,. \tag{3.4}$$

When $r=s$, (3.3) follows from (2.13)–(2.14); indeed, (2.14) yields the following system of equations for the exponents $\delta_{\rho\alpha}$ of (3.3),

$$\sum_{\alpha=1}^{r} \delta_{\rho\alpha} \begin{bmatrix} c_{\alpha 1} \\ c_{\alpha 2} \\ \vdots \\ c_{\alpha r} \end{bmatrix} = - \begin{bmatrix} c_{\rho 1} \\ c_{\rho 2} \\ \vdots \\ c_{\rho r} \end{bmatrix} \qquad (\rho = [r+1], \dots, p) \tag{3.5}$$

In like manner, (2.14) yields the following system of equations for the exponents $\gamma_{k\alpha}$ of (3.2),

$$\sum_{\alpha=1}^{r} \gamma_{k\alpha} \begin{bmatrix} c_{\alpha 1} \\ c_{\alpha 2} \\ \vdots \\ c_{\alpha r} \end{bmatrix} = - \begin{bmatrix} b_{k1} \\ b_{k2} \\ \vdots \\ b_{kr} \end{bmatrix} \qquad (k = 1, \dots, m) \tag{3.6}$$

The converse of Theorem 2 readily follows.

For ease in following discussions, it is helpful to introduce,

Definition 2: *The absolute invariants $\{\Pi_j, \tilde{\pi}_k\}$ of Theorem 2 are termed* normalized *variables.* See [12] for a like usage.

* It is assumed that $p>s$. For the special case $p=s$, no absolute invariants are determined *solely* from the physical variables.

It is illuminating to view Theorem 1 within the context of the results provided by Theorem 2. Thus, when $r=s$, (2.8) of Theorem 1 may be rewritten as,

$$Z_j[Y_1]^{\lambda_{j1}}\dots[Y_r]^{\lambda_{jr}}=F_j(\hat{\pi}_1,\dots,\hat{\pi}_m;\tilde{\pi}_{r+1},\dots,\tilde{\pi}_p)\,. \tag{3.7}$$

Clearly, the number of absolute invariants $\tilde{\pi}$ involving only *physical* variables is $[p-r]$; so in this sense it might be said that the number of physical variables has been reduced from p in the original relationship I_j to $[p-r]$ in F_j. On the other hand, the number of *normalized* independent variables $\hat{\pi}$ in F_j is *precisely* the same as the number of independent variables in I_j; namely, m. As a conclusion, therefore, when Theorem 1 is applied via a group (2.5)–(2.7) for which $r=s$, the outcome of the application can only lead to a reduction in the number of physical variables, and *cannot* lead to a reduction in the number of independent variables.

Further conclusions may often be elicited upon inspection of the dimensional matrix associated with (2.5)–(2.7). To develop this point further, rewrite (3.7) as,

$$Z_j=[Y_1]^{-\lambda_{j1}}\dots[Y_r]^{-\lambda_{jr}}F_j(\hat{\pi}_1,\dots,\tilde{\pi}_p)\,. \tag{3.8}$$

Thus, Z_j varies directly as, say, $[Y_r]^{-\lambda_{jr}}$ whenever the variables $\hat{\pi}_k$, $\tilde{\pi}_\rho$ are independent of Y_r. To determine when such behavior may be anticipated, consider (3.2) and (3.3). From (3.2) it follows that $\hat{\pi}_k$ is independent of Y_r whenever $\gamma_{kr}=0$; likewise, (3.3) reveals that $\tilde{\pi}_\rho$ is independent of Y_r whenever $\delta_{\rho r}=0$. Thus, with (3.5) and (3.6) it may be concluded that Z_j varies directly as $[Y_r]^{-\lambda_{jr}}$ whenever the first $[r-1]$ rows of C span the matrix derived from BC upon deleting row r of C. Clearly, this reasoning can be extended.

The foregoing discussion has focused upon groups (2.5)–(2.7) with $r=s$, the equally important case $r>s$ will now be treated.

Theorem 3: *If, and only if* $r>s$, *the set of* $[n+m+p-r]$ *independent absolute invariants required by Theorem* 1 *may be obtained in the form,*

$$\Pi_j=Z_j[X_\varepsilon]^{A_{j\varepsilon}}\dots[X_m]^{A_{jm}}[Y_1]^{\lambda_{j1}}\dots[Y_s]^{\lambda_{js}}\qquad(j=1,\dots,n) \tag{3.9}$$

$$\hat{\pi}_\sigma=X_\sigma[X_\varepsilon]^{\Gamma_{\sigma\varepsilon}}\dots[X_m]^{\Gamma_{\sigma m}}[Y_1]^{\gamma_{\sigma1}}\dots[Y_s]^{\gamma_{\sigma s}}\qquad(\sigma=1,\dots,[m+s-r]) \tag{3.10}$$

$$\tilde{\pi}_\rho=Y_\rho[Y_1]^{\delta_{\rho1}}\dots[Y_s]^{\delta_{\rho s}}\qquad(\rho=[s+1],\dots,p) \tag{3.11}$$

wherein $\varepsilon\equiv[m+s-r+1]\leq m$.

Eq. (3.9) is readily established via (2.15)–(2.16) upon utilizing the assumed condition that when $r>s$, the last $[r-s]$ rows of the matrix B plus the first s rows of the matrix C are linearly independent. Thus, (2.16) yields the following system of equations for the exponents of (3.9),

$$\sum_{\alpha=\varepsilon}^{m}A_{j\alpha}\begin{bmatrix}b_{\alpha1}\\b_{\alpha2}\\\vdots\\b_{\alpha r}\end{bmatrix}+\sum_{\omega=1}^{s}\lambda_{j\omega}\begin{bmatrix}c_{\omega1}\\c_{\omega2}\\\vdots\\c_{\omega r}\end{bmatrix}=-\begin{bmatrix}a_{j1}\\a_{j2}\\\vdots\\a_{jr}\end{bmatrix}\qquad(j=1,\dots,n). \tag{3.12}$$

When $r>s$, (3.11) follows from (2.13)–(2.14); indeed, (2.14) yields the following system of equations for the exponents of (3.11),

$$\sum_{\omega=1}^{s}\delta_{\rho\omega}\begin{bmatrix}c_{\omega1}\\c_{\omega2}\\\vdots\\c_{\omega r}\end{bmatrix}=-\begin{bmatrix}c_{\rho1}\\c_{\rho2}\\\vdots\\c_{\rho r}\end{bmatrix}\qquad(\rho=[s+1],\dots,p)\,. \tag{3.13}$$

In like manner, (2.14) yields the following system of equations for the exponents of (3.10),

$$\sum_{\alpha=\varepsilon}^{m} \Gamma_{\sigma\alpha} \begin{bmatrix} b_{\alpha 1} \\ b_{\alpha 2} \\ \vdots \\ b_{\alpha r} \end{bmatrix} + \sum_{\omega=1}^{s} \gamma_{\sigma\omega} \begin{bmatrix} c_{\omega 1} \\ c_{\omega 2} \\ \vdots \\ c_{\omega r} \end{bmatrix} = - \begin{bmatrix} b_{\sigma 1} \\ b_{\sigma 2} \\ \vdots \\ b_{\sigma r} \end{bmatrix} \qquad (\sigma = 1, \ldots, [m+s-r]) . \tag{3.14}$$

The converse of Theorem 3 readily follows.

The following discussions are eased by the introduction of,

Definition 3: *$\hat{\pi}_\sigma$ is termed a* similarity *variable whenever at least one of the exponents $\Gamma_{\sigma\alpha}$ $(\alpha=\varepsilon, \ldots, m)$ is non-zero; and is termed a* normalized *variable whenever each of the exponents $\Gamma_{\sigma\alpha}$ is zero.*

Like considerations also follow for the invariants Π_j of Theorem 3.

It is illuminating to view Theorem 1 within the context of the results provided by Theorem 3. Thus, when $r > s$, (2.8) of Theorem 1 may be rewritten as,

$$\{Z_j [X_\varepsilon]^{\Lambda_{j\varepsilon}} \ldots [X_m]^{\Lambda_{jm}} [Y_1]^{\lambda_{j1}} \ldots [Y_s]^{\lambda_{js}}\} = F_j(\hat{\pi}_1, \ldots, \hat{\pi}_{m+s-r}, \tilde{\pi}_{s+1}, \ldots, \tilde{\pi}_p) . \tag{3.15}$$

It is clear that the number of independent variables $\hat{\pi}$ in F_j is fewer than the number of independent variables in the original relationship I_j. As a conclusion, therefore, when Theorem 1 is applied via a group (2.5)–(2.7) for which $r > s$, the outcome of the application is a reduction in the number of independent variables.

As for the case $r = s$, further conclusions may often be elicited for the case $r > s$ upon inspection of the dimensional matrix associated with (2.5)–(2.7). To develop this point further, rewrite (3.15) as,

$$Z_j = \{[X_\varepsilon]^{-\Lambda_{j\varepsilon}} \ldots [X_m]^{-\Lambda_{jm}} [Y_1]^{-\lambda_{j1}} \ldots [Y_s]^{-\lambda_{js}}\} F_j(\hat{\pi}_1, \ldots, \hat{\pi}_p) . \tag{3.16}$$

Thus, Z_j varies directly as, say, $[X_\varepsilon]^{-\Lambda_{j\varepsilon}}$ whenever the variables $\hat{\pi}_\sigma$ are independent of X_ε. To determine when such behavior may be anticipated, consider (3.10). From (3.10) it follows that $\hat{\pi}_\sigma$ is independent of X_ε whenever $\Gamma_{\sigma\varepsilon}=0$. Thus, with (3.14) it may be concluded that Z_j varies directly as $[X_\varepsilon]^{-\Lambda_{j\varepsilon}}$ whenever the last $[r-s-1]$ rows of the matrix B plus the first s rows of C span the matrix derived from BC upon deleting the row corresponding to X_ε. Clearly, this reasoning can be extended.

The results of the present article are illustrated in § 5–§ 7 by application to a number of typical engineering problems.

4. On the Sedov Self-Similarity Criterion

In formulating a conventional dimensional approach to one-dimensional, unsteady gas flows, Sedov [16, pp. 146–148] adopts dimensional formulae corresponding to the scale change equations,

$$\begin{aligned} \bar{Z}_j &= M^{a_{j1}} L^{a_{j2}} \tau^{a_{j3}} Z_j \qquad (j=1, \ldots, n) \\ \bar{x} &= M^0 L^{+1} \tau^0 x \\ \bar{t} &= M^0 L^0 \tau^{+1} t \\ \bar{Y}_1 &= M^0 L^{c_{12}} \tau^{c_{13}} Y_1 \\ \bar{Y}_e &= M^{c_{e1}} L^{c_{e2}} \tau^{c_{e3}} Y_e \qquad (e=2, \ldots, p) \end{aligned} \tag{4.1}$$

where $\{c_{12}, c_{13}, c_{21}\}$ are assumed to be non-zero; x denotes a position coordinate, t symbolizes time, the Y's represent the associated physical variables (cf. (2.3)–(2.4)).

Equations (4.1) have the same form as the class of transformation groups under consideration: (2.5)–(2.7). It is illuminating to inquire into the nature of the gas flows for which Theorem 3 is applicable.

The ranks of the matrices C and BC are required to determine when Theorem 3 can be applied. From (4.1) it follows that BC has the form,

$$BC: \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 0 & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ \vdots & \vdots & \vdots \\ c_{p1} & c_{p2} & c_{p3} \end{bmatrix}$$

with the matrix C corresponding to the last p rows. In light of the requirement that $c_{21} \neq 0$, the rank r of BC is three: $r=3$; furthermore, because c_{12} and c_{13} are also presumed to be non-zero, the rank s of the matrix BC is at least two: $s \geqq 2$.

Theorem 3 is applicable only when the rank r of the matrix BC is greater in value than the rank s of the matrix $C: r > s$. Thus, it is necessary that $s=2$; that is, of the set of physical variables, only two may be dimensionally independent. For $r=3$, $s=2$ a similarity type result evolves: With Theorem 3,

$$\begin{aligned} \Pi_j &= Z_j[t]^{\Lambda_{j2}}[Y_1]^{\lambda_{j1}}[Y_2]^{\lambda_{j2}} \qquad (j=1, \ldots, n) \\ \hat{\pi}_1 &= x[t]^{\Gamma_{12}}[Y_1]^{\gamma_{11}}[Y_2]^{\gamma_{12}} \\ \tilde{\pi}_\rho &= Y_\rho[Y_1]^{\delta_{\rho 1}}[Y_2]^{\delta_{\rho 2}} \qquad (\rho = 3, \ldots, p) \end{aligned} \tag{4.2}$$

wherein,

$$\Lambda_{j2}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \lambda_{j1}\begin{bmatrix} 0 \\ c_{12} \\ c_{13} \end{bmatrix} + \lambda_{j2}\begin{bmatrix} c_{21} \\ c_{22} \\ c_{23} \end{bmatrix} = -\begin{bmatrix} a_{j1} \\ a_{j2} \\ a_{j3} \end{bmatrix} \tag{4.3}$$

$$\Gamma_{12}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \gamma_{11}\begin{bmatrix} 0 \\ c_{12} \\ c_{13} \end{bmatrix} + \gamma_{12}\begin{bmatrix} c_{21} \\ c_{22} \\ c_{23} \end{bmatrix} = -\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{4.4}$$

$$\delta_{\rho 1}\begin{bmatrix} 0 \\ c_{12} \\ c_{13} \end{bmatrix} + \delta_{\rho 2}\begin{bmatrix} c_{21} \\ c_{22} \\ c_{23} \end{bmatrix} = -\begin{bmatrix} c_{\rho 1} \\ c_{\rho 2} \\ c_{\rho 3} \end{bmatrix} \tag{4.5}$$

Thus, $\gamma_{12}=0$, $\gamma_{11}=-1/c_{12}$, $\Gamma_{12}=c_{13}/c_{12}$; and π_1 is a *similarity* variable.

The foregoing development may be summarized by noting that to achieve a similarity type result via (4.1), the set of physical parameters associated with a given flow may include only two members with independent dimensions. This summary exactly corresponds to the well-known Sedov self-similarity criterion, [16, p. 148]; (see [17], [18], [19] for recent applications of the criterion). It should be noted, however, that the Sedov criterion is tied to the particular class of transformation groups (4.1); and is therefore limited in scope. On the other hand, Theorem 3 can be applied with any group of the form (2.5)–(2.7) for which $r > s$, including ones of the form (4.1). Thus, Theorem 3 constitutes a generalized formulation for similarity which includes the Sedov criterion, but which may be invoked in cases wherein (4.1) is not appropriate—e.g., see the illustration of § 7.

5. Flow Near a Wall in Motion.

The first illustration to be considered is the incompressible flow problem introduced in § 2, represented by,

$$u_t - \nu u_{yy} = 0 \quad \text{(momentum)} \tag{2.1}$$

$$\begin{cases} u \to 0 & \text{as } t \to 0 \quad \text{when } y \geqq 0 \\ u \to 0 & \text{as } y \to \infty \quad \text{when } t \geqq 0 \\ u \to U & \text{as } y \to 0 \quad \text{when } t > 0 \end{cases} \tag{2.2}$$

Initially, the problem is analyzed from a conventional dimensional viewpoint—i.e., by application of the two-parameter group (2.4). Then, the generalized dimensional analysis approach described in [14] is invoked to establish a somewhat stronger conclusion via Theorem 3.

The governing equations (2.1)–(2.2) are easily shown to be invariant in form under the two-parameter group (2.4); and this suggests that the solution $u = I(y, t; U, v)$ is also invariant in form under (2.4), [1], [14]. Therefore, Theorem 1 can be invoked to establish an equivalent relationship in fewer variables. Furthermore, the appropriate Theorem of §3 may be utilized to develop the absolute invariants required by Theorem 1.

To determine which of the theorems of §3 is appropriate for the present application, the ranks of the matrices C and BC associated with (2.4) must be evaluated; thus, consider,

$$BC: \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \hline 2 & -1 \\ 1 & -1 \end{bmatrix}$$

wherein the matrix C corresponds to the last two rows. By inspection, $r = s = 2$; consequently Theorem 2 is appropriate for establishing the absolute invariants of (2.4).

The following set of *normalized* variables can be shown to evolve via Theorem 2,

$$\{\Pi = u/U, \quad \hat{\pi}_1 = y/[v/U], \quad \hat{\pi}_2 = t/[v/U^2]\}. \tag{5.1}$$

Elementary chain-rule operations are sufficient to show that the governing equations (2.1)–(2.2) may be rewritten in terms of only the normalized variables (5.1), [14]. Consequently, it follows that in accord with the conclusion of Theorem 1, $\Pi = F(\hat{\pi}_1, \hat{\pi}_2)$.

In summary, then, an application of Theorem 1 within the context of a traditional approach to dimensions has led to the *normalized* variables (5.1). The generalized approach described in [14] will now be coupled with Theorem 3 to deduce *similarity* variables.

Instead of beginning this phase of the analysis with the transformations (2.4), the present analysis is initiated with transformations of the form,

$$\begin{aligned} &\bar{u} = A_1 u\,, && \text{(dependent variable)} \\ &\bar{y} = A_2 y\,,\ \bar{t} = A_3 t && \text{(independent variables} \\ &\bar{v} = A_4 v\,,\ \bar{U} = A_5 U && \text{(physical variables)} \end{aligned} \tag{5.2}$$

The A's of (5.2) must necessarily be interrelated in order to meet the requirement of Theorem 1 that (2.1)–(2.2) be invariant in form. Indeed, this requirement is met whenever the following relationships exist among the A's, $A_5 = A_1$, $A_4 = A_2^2 A_3^{-1}$; or in other words, (2.1)–(2.2) is invariant in form under the three-parameter group,

$$\begin{aligned} \bar{u} &= A_1^{+1} A_2^0 A_3^0 u \quad & \bar{y} &= A_1^0 A_2^{+1} A_3^0 y \quad & \bar{v} &= A^0 A_2^{+2} A_3^{-1} v \\ & & \bar{t} &= A_1^0 A_2^0 A_3^{+1} t & \bar{U} &= A_1^{+1} A_2^0 A_3^0 U \end{aligned} \tag{5.3}$$

see [14] for further detail.*

To determine which of the two theorems of §3 is appropriate for establishing absolute invariants for (5.3) requires that the ranks of the associated matrices C and BC be evaluated; thus,

* The traditional dimensional transformation group (2.4), a two-parameter group, is *included* in the generalized dimensional transformation group (5.3), a three-parameter group. This may be verified by substituting $A_1 = L^{+1}\tau^{-1}$, $A_2 = L$, $A_3 = \tau$ into (5.3).

$$BC: \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hdashline 0 & 2 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

wherein the matrix C corresponds to the last two rows.

By inspection, $r=3$ and $s=2$; consequently, Theorem 3 is appropriate for establishing absolute invariants. Indeed,

$$\Pi = u/U\,, \quad \hat{\pi} = y/[\nu t]^{\frac{1}{2}}\,. \tag{5.4}$$

Clearly, $\hat{\pi}$ is a *similarity* variable. Furthermore, recalling the discussion associated with (3.16), the absence of the physical variable U from $\hat{\pi}$ might well have been predicted by observing that the last row of BC, which corresponds to U is independent of the remaining rows.

The change of variables provided by (5.4) is well known [11], and leads to a solution for the system (2.1)–(2.2) in terms of the *error* function,

$$\Pi = 1 - \operatorname{erf}(\hat{\pi}/2) \tag{5.5}$$

In conclusion, it is noted that the result (5.5) is achieved by application of Theorem 3 via the generalized dimensional analysis approach, but is not established by means of a traditional dimensional approach.

6. The Blasius Problem.

The next illustration to be considered concerns the so-called Blasius problem of two-dimensional, incompressible flow past a flat plate. The problem is governed by the following equations,

$$\begin{aligned} uu_x + vu_y &= \nu u_{yy} \quad \text{(momentum)} \\ u_x + v_y &= 0 \qquad\quad \text{(continuity)} \end{aligned} \tag{6.1}$$

subject to,

$$\begin{aligned} &u = v = 0 \ \text{ at } \ y = 0 \\ &u \to U \ \text{ as } \ y \to \infty \quad \text{and} \quad \text{as } \ x \to 0 \end{aligned} \tag{6.2}$$

where the velocity component normal to the plate is denoted by $v(x, y)$; $u(x, y)$ symbolizes the velocity component parallel to the plate, and U is its constant-valued limit as the normal distance y approach infinity; ν represents the constant kinematic viscosity of the fluid; and x denotes distance along the plate as measured from the leading edge, [16].

The solution to (6.1)–(6.2) is well known, being determined via the change of variables (6.3),

$$u/U = F_1[y(U/\nu x)^{\frac{1}{2}}], \quad v(x/\nu U)^{\frac{1}{2}} = F_2[y(U/\nu x)^{\frac{1}{2}}] \tag{6.3}$$

for instance, see Sedov [16]. In his analysis of the Blasius problem Sedov shows that the change of variables (6.3) *cannot* be deduced via an application of a conventional dimensional approach *without* introducing an auxiliary discussion. The problem of establishing the variables of (6.3) will now be investigated by means of the theorems of §3.

A conventional application of dimensional analysis to the Blasius problem would utilize dimensional formulae corresponding to the scale change equations (6.4),

$$\begin{aligned} &\bar{u} = L^{+1}\tau^{-1}u, \ \bar{v} = L^{+1}\tau^{-1}v \quad \text{(dependent variables)} \\ &\bar{x} = L^{+1}\tau^{0}x, \ \bar{y} = L^{+1}\tau^{0}y \qquad \text{(independent variables)} \\ &\bar{\nu} = L^{+2}\tau^{-1}\nu, \ \bar{U} = L^{+1}\tau^{-1}U \ \text{(physical variables)} \end{aligned} \tag{6.4}$$

Equations (6.4) constitute a two-parameter group, with the scale factors L and τ playing the role of the group parameters. Moreover, as may be shown, (6.1)–(6.2) is invariant in form under (6.4).

To determine which of the two theorems of §3 is appropriate for establishing absolute invariants for (6.4) requires that the ranks of the associated matrices C and BC be evaluated; thus,

$$BC: \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ \hdashline 2 & -1 \\ 1 & -1 \end{bmatrix}$$

where C corresponds to the last two rows. For BC, $r=s=2$. It follows that Theorem 2, rather than Theorem 3 is appropriate; therefore, the similarity result (6.3) cannot evolve via (6.4). So, indeed, a conventional dimensional approach is not effective in establishing (6.3).

Next, the generalized dimensional analysis procedure is invoked. Upon initiating the analysis with a general transformation of the form $\bar{u}=A_1u$, $\bar{\nu}=A_2\nu$, $\bar{x}=A_3x$, etc., it is readily shown the A's must be interrelated in order for (6.1)–(6.2) to be invariant in form. Thus, as may be shown, (6.1)–(6.2) is invariant in form under the three-parameter group (6.5),

$$\begin{cases} \bar{u}=A_1^{+1}A_2^0A_3^0u, & \bar{v}=A_1^{+\frac{1}{2}}A_2^{\frac{1}{2}}A_3^{-\frac{1}{2}}v \\ \bar{x}=A_1^0A_2^0A_3^{+1}x, & \bar{y}=A_1^{-\frac{1}{2}}A_2^{\frac{1}{2}}A_3^{\frac{1}{2}}y \\ \bar{\nu}=A_1^0A_2^{+1}A_3^0\nu, & \bar{U}=A_1^{+1}A_2^0A_3^0U \end{cases} \tag{6.5}$$

The matrix BC associated with (6.5) has the form,

$$BC: \begin{bmatrix} 0 & 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \hdashline 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

By inspection, $r=3$ and $s=2$. Therefore, Theorem 3 is applicable; and the variables of (6.3) can be shown to evolve naturally upon application of Theorem 3.

In summary, the Blasius problem illustrates once again the superiority of generalized dimensional analysis over the traditional dimensional approach, by achieving similarity variables rather than normalized variables. Too, the Blasius problem illustrates the utility of the theorems of §3 for determining the general outcome of an analysis with a particular group, without the need for directly establishing the absolute invariants of the group—and this point is given further emphasis in the following discussion.

7. Extrusion with Sublimation.

The steady, laminar incompressible boundary layer on a moving continuous flat surface with sublimation (or condensation) has been studied in [20], wherein the objective is to determine the velocity, temperature and concentration profiles within the boundary layer. The governing differential equations are given by,

$$\begin{aligned} & u_x+v_y=0 && \text{(continuity)} \\ & uu_x+vu_u-\nu u_{yy}=0 && \text{(momentum)} \\ & uT_x+vT_y-\alpha T_{yy}=0 && \text{(energy)} \\ & uC_x+vC_y-DC_{yy}=0 && \text{(diffusion)} \end{aligned} \tag{7.1}$$

where u denotes the velocity component parallel to the plate, and v signifies the velocity component normal to the plate. Position along the plate is designated by x, while position normal to the plate is designated by y. The constant transport coefficients for momentum, energy and diffusion of species are symbolized, respectively by ν, α, and D.

The objective of the present discussion is to establish via Theorem 3 the number of physical variables with independent dimensions that may be introduced (via the boundary conditions), and still achieve a similarity type result.

The generalized dimensional analysis approach of [14] may be invoked to show that (7.1) is invariant in form under the transformations of (7.2),

$$\left\{\begin{array}{ll} \bar{T} = A_1^1A_2^0A_3^0A_4^0A_5^0T, \quad \bar{C} = A_1^0A_2^1A_3^0A_4^0A_5^0C & \text{(dependent variables)} \\ \bar{u} = A_1^0A_2^0A_3^1A_4^0A_5^0u, \quad \bar{v} = A_1^0A_2^0A_3^1A_4^{-1}A_5^1v & \end{array}\right.$$
$$\left\{\ \bar{x} = A_1^0A_2^0A_3^0A_4^1A_5^0x, \quad \bar{y} = A_1^0A_2^0A_3^0A_4^0A_5^1y \quad \text{(independent variables)}\right. \tag{7.2}$$
$$\left\{\begin{array}{ll} \bar{\nu} = A_1^0A_2^0A_3^1A_4^{-1}A_5^2\nu, \quad \bar{\alpha} = A_1^0A_2^0A_3^1A_4^{-1}A_5^2\alpha, & \text{(physical variables)} \\ \bar{D} = A_1^0A_2^0A_3^1A_4^{-1}A_5^2D & \end{array}\right.$$

The dimensional matrix associated with (7.2) is,

$$\left[\begin{array}{lrrrrr} T: & 1 & 0 & 0 & 0 & 0 \\ C: & 0 & 1 & 0 & 0 & 0 \\ u: & 0 & 0 & 1 & 0 & 0 \\ v: & 0 & 0 & 1 & -1 & 1 \\ \hline x: & 0 & 0 & 0 & 1 & 0 \\ y: & 0 & 0 & 0 & 0 & 1 \\ \hline \nu: & 0 & 0 & 1 & -1 & 2 \\ \alpha: & 0 & 0 & 1 & -1 & 2 \\ D: & 0 & 0 & 1 & -1 & 2 \end{array}\right]$$

Since the rank r is five, Theorem 3 indicates that for a similarity type result there can be *at most* four physical variables with independent dimensions; or in other words, in addition to ν, α, D there can be *at most* three physical variables with independent dimensions.*

In light of the foregoing conclusion it is pertinent to point out that [20] presents a similarity type solution for (7.1) with boundary conditions involving precisely three additional physical variables with independent dimensions. These boundary conditions are,

$$\begin{array}{llll} \text{at } y=0: & u=U, & T=T_w, & C=C_w, \quad v=V_0/\sqrt{x} \\ \text{as } y\to\infty: & u\to 0, & T\to T_\infty, & C\to C_\infty \end{array} \tag{7.3}$$

wherein U, T_w, T_∞, C_w, C_∞, V_0 are physical variables.

The dimensional matrix associated with the physical variables of (7.1) and (7.3) is,

$$\left[\begin{array}{lrrrrr} T_w: & 1 & 0 & 0 & 0 & 0 \\ T_\infty: & 1 & 0 & 0 & 0 & 0 \\ C_w: & 0 & 1 & 0 & 0 & 0 \\ C_\infty: & 0 & 1 & 0 & 0 & 0 \\ U: & 0 & 0 & 1 & 0 & 0 \\ V_0: & 0 & 0 & 1 & -\frac{1}{2} & 1 \\ \nu: & 0 & 0 & 1 & -1 & 2 \\ \alpha: & 0 & 0 & 1 & -1 & 2 \\ D: & 0 & 0 & 1 & -1 & 2 \end{array}\right]$$

* It may be shown that the traditional dimensions are included in (7.2). Thus, upon letting $A_1=\theta$, $A_2=ML^{-3}$, $A_3=L\tau^{-1}$, $A_4=L$, and $A_5=L$ and substituting these into (7.2), the traditional dimensional transformations are obtained. The dimensional matrix, BC, associated with the independent and physical variables for *traditional dimensions* has rank $r=4$. The rank of the C matrix is $s=4$.

The rank s of this matrix is four. Therefore, with Theorem 3, the absolute invariants are obtained as similarity variables:*

$$\Pi_1 = T^{+1}T_w^{-1},\ \Pi_2 = C^{+1}C_w^{-1},\ \Pi_3 = u^{+1}U^{-1},\ \Pi_4 = vx^{\frac{1}{2}}V_0^{-1}$$

$$\hat{\pi}_1 = x^{+1}y^{-2}\alpha^{+2}V_0^{-2} \tag{7.4}$$

$$\tilde{\pi}_1 = T_w^{+1}T_\infty^{-1},\ \tilde{\pi}_2 = C_w^{+1}C_\infty^{-1},\ \tilde{\pi}_3 = v^{+1}\alpha^{-1},\ \tilde{\pi}_4 = v^{+1}D^{-1},\ \tilde{\pi}_5 = V_0^{+2}v^{-1}U^{-1},$$

In conclusion, it is noted that this problem clearly reveals the effectiveness of Theorem 3 as a generalization of Sedov's similarity criterion.

8. Closure.

In this paper a careful distinction among the variables appearing in a given set of governing equations is shown to be instrumental for utilizing most effectively the generalized dimensional analysis formulation presented in [14]. In particular, the delineation of three distinct categories for the variables as *dependent*, *independent*, or *physical* naturally leads to a straightforward test for determining if a similarity type result can be achieved for a given set of governing equations, via a given *dimensional* group with the form (2.5)–(2.7). Indeed, a review of the principal results reported (Theorems 2, 3) reveals that to apply the test, it is only necessary to determine the ranks of two matrices associated with the group being considered. Additionally, it has been shown that Theorem 3 constitutes a generalization of Sedov's self-similarity criterion.

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* On the other hand, for the case of traditional dimensions $r = s = 4$; so, Theorem 2 is applicable and the absolute invariants are obtained as *normalized* variables.